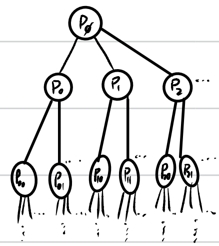
Descriptive Set Theory
Lecture 25

wlog assume $P_s \supseteq P_{s^\frown n}$

Operation $\mathcal{A}$ (recall). Operation $\mathcal{A}$ applies to a sequence $(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of a Polish space X, yielding the set

$$\mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}} := \underbrace{\bigcup_{x\in\mathbb{N}^{\mathbb{N}}}}_{\exists x\in\mathbb{N}^{\mathbb{N}}} \underbrace{\bigcap_{n}}_{\forall n} P_{x|_n}.$$



The typical application was: let $f:\mathbb{N}^{\mathbb{N}}\to X$ be continuous and $P_s := f([s])$. We proved that $f(\mathbb{N}^{\mathbb{N}}) = \mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}} = \mathcal{A}(\overline{P_s})_{s\in\mathbb{N}^{<\mathbb{N}}}$. This implies $\Sigma^1_1 \subseteq \mathcal{A}\Pi^0_1$. It's clear that $\mathcal{A}\Sigma^1_1 \subseteq \Sigma^1_1$, so

$$\Sigma^1_1 = \mathcal{A}\Pi^0_1 = \mathcal{A}\Sigma^1_1.$$

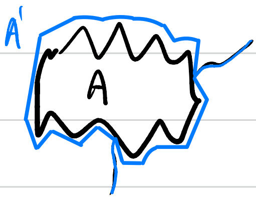
Today we show that the classes of Baire meas. and univ. meas. sets are closed under operation $\mathcal{A}$. Since Π^0_1 belongs to these classes, so does $\Sigma^1_1 = \mathcal{A}\Pi^0_1$, so analytic (and hence also coanalytic) sets are Baire meas. and universally meas.

Def. For a σ-algebra $\mathcal{S}$ of subsets of X, call a set $A\subseteq X$ **$\mathcal{S}$-small** if $\mathrm{Powerset}(A)\subseteq\mathcal{S}$; otherwise it's **$\mathcal{S}$-large**.
$\mathcal{S}$-small sets form a σ-ideal.

Examples. Meager sets for BM σ-alg, and μ-null sets for μ-meas. σ-alg.

Recall that for BM and Lebesgue meas. σ-algebras, we had a way of extracting the largest measurable subset A' (up to small sets) of a given set $A \subseteq X$. For BM, this was $\mathcal{U}(A) \cap A$, and for Lebesgue meas., this was $\mathcal{D}(A) \cap A$. Here we define the dual notion to this, namely smallest meas. superset.

<u>Def.</u> Let $\mathcal{S}$ σ-alg on a set X. We say that a set A' is an **$\mathcal{S}$-envelope** of a set $A \subseteq X$ if $A' \supseteq A$, $A' \in \mathcal{S}$, and any $\mathcal{S}$-meas subset of $A' \setminus A$ is $\mathcal{S}$-small. In other words,

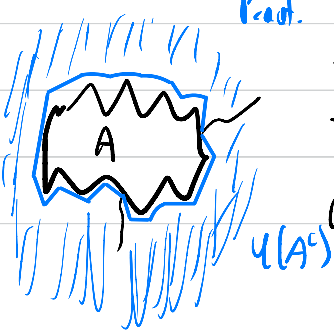


A' is the smallest (up to $\mathcal{S}$-small) $\mathcal{S}$-meas. superset of A. Say that $\mathcal{S}$ **admits envelopes** if every $A \subseteq X$ has an **$\mathcal{S}$-envelope**.

<u>Obs.</u> Envelopes are unique up to small sets.

<u>Examples.</u> (a) BM σ-alg admits envelopes.

Proof. Given $A \subseteq X$, X Polish, we know that $\mathcal{U}(A^c) \cap A^c$ is the largest BM subset of A^c, up to meager sets. Thus, $(\mathcal{U}(A^c) \cap A^c)^c$ is the smallest BM superset of A, up to meager. $\square$



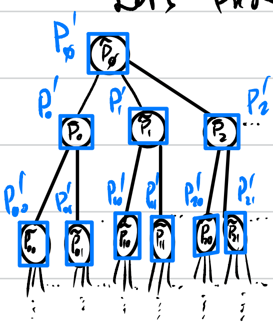
(b) μ-meas σ-alg admits envelopes, for any finite Borel meas. μ.

Proof. By throwing out the ctbl set of atoms, may assume μ is nonatomic, so WLOG μ = Leb meas. on $[0,1)$.

For any $A \subseteq [0,1)$, take $A' := (\mathcal{D}(A^c) \cap A^c)^c$. $\square$

Theorem (Szpilrajn-Marczewski). If a σ-alg $\mathcal{S}$ on a set X admits envelopes, then it is closed under operation $\mathcal{A}$.

Proof. Let $(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}} \subseteq \mathcal{S}$. If we had that each $P_s = \bigcup_n P_{s^\frown n}$, then $\mathcal{A}(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}} = P_\emptyset \in \mathcal{S}$, so we'd be done. We'll reduce to the case where $P_s = \bigcup_n P_{s^\frown n}$ up to an $\mathcal{S}$-small set, which is enough because throwing out ctbly many $\mathcal{S}$-small sets from the space X doesn't affect the $\mathcal{S}$-meas. of $\mathcal{A}(P_s)_s$.



Consider $\tilde{P}_s := \mathcal{A}(P_t)_{s \subseteq t \in \mathbb{N}^{<\mathbb{N}}}$. Note that $\tilde{P}_s \subseteq P_s$ and $\tilde{P}_s = \bigcup_n \tilde{P}_{s^\frown n}$, so $\mathcal{A}(\tilde{P}_s)_{s \in \mathbb{N}^{<\mathbb{N}}} = \tilde{P}_\emptyset = \mathcal{A}(P_s)_s$. But we don't know if $\tilde{P}_\emptyset$ is $\mathcal{S}$-meas, so we need something in between $P_s \supseteq ? \supseteq \tilde{P}_s$.

Let $P'_s := \mathrm{env}(\tilde{P}_s) \cap P_s$, where $\mathrm{env}(\tilde{P}_s)$ is an $\mathcal{S}$-env for $\tilde{P}_s$. Thus $\mathcal{A}(P_s)_s \supseteq \mathcal{A}(P'_s)_s \supseteq \mathcal{A}(\tilde{P}_s)_s = \mathcal{A}(P_s)_s$, so $\mathcal{A}(P_s)_s = \mathcal{A}(P'_s)_s$.

Note that $\bigcup_n P'_{s^\frown n}$ is an $\mathcal{S}$-env. for $\bigcup_n \tilde{P}_{s^\frown n} = \tilde{P}_s$, so by uniqueness, P'_s and $\bigcup_n P'_{s^\frown n}$ are equal mod $\mathcal{S}$-small, which is what we needed. $\square$

Cor. Analytic (and hence also coanalytic) sets are Baire meas. and univ. meas.

Projective Hierarchy.

We know that $\Sigma^1_1 = \exists^{\mathbb{N}^\mathbb{N}} \mathcal{B}$ and we continue:

$$\begin{aligned}
\Sigma^1_{n+1} &:= \exists^{\mathbb{N}^\mathbb{N}} \Pi^1_n \\
\Pi^1_n &:= \neg \Sigma^1_n \\
\Delta^1_n &:= \Sigma^1_n \wedge \Pi^1_n .
\end{aligned}$$

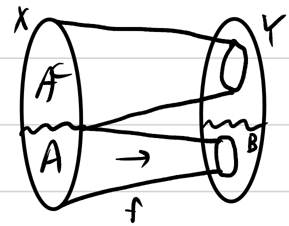
$$\underbrace{\begin{array}{ccccccccccccccc}
 & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & & & \Sigma^1_n & & \\
\text{Borel} = \Delta^1_1 & \subsetneq & & \subsetneq & \Delta^1_2 & \subsetneq & & \subsetneq & \Delta^1_3 & \subsetneq & \cdots\cdots & \Delta^1_n \subsetneq & & \subsetneq & \Delta^1_{n+1} \cdots \\
 & \subsetneq & \Pi^1_1 & \subsetneq & & \subsetneq & \Pi^1_2 & \subsetneq & & \subsetneq & & \subsetneq & \Pi^1_n & \subsetneq &
\end{array}}_{\text{projective hierarchy, length } \omega}$$

The existence of universal sets and strict containments follow as before by diagonalization.
We won't study these sets because questions about them are indep. from ZFC.

Complete sets.

Def. For sets $A \subseteq X$ and $B \subseteq Y$, X, Y Polish, a function $f: X \to Y$ is called a reduction of A to B if $\forall x \in X$,

$$x \in A \iff f(x) \in B,$$



i.e. $f^{-1}(B) = A$. We say that A is Wadge reducible to B if there is a continuous reduction of A to B.

We write $(A, X) \leq_W (B, Y)$.

Def. Let Y be Polish and Γ be a class of sets (Π^0_7, Σ^1_1, etc).

We say that $B \subseteq Y$ is Γ-hard (resp. Γ-complete) if $\forall$ subset $A \subseteq X$ in a 0-dim Polish X, $(A, X) \leq_W (B, Y)$ (resp. and $B \in \Gamma$).

Note. If set is Γ-hard/complete, then its complement is $\neg\Gamma$-hard/comp.

Remark. We require the domain be 0-dim to remove the top obstructions (connectedness) to building continuous reductions.

Note that if a set is say Σ^0_7-hard, then it is not Π^0_7 (otherwise, every Σ^0_7 set would be a continuous preimage of a Π^0_7 set hence $\Sigma^0_7 \subseteq \Pi^0_7$, which is false). Turns out that this is the only obstruction to being in Π^0_7.

Theorem (Wadge). Let Y be a 0-dim Polish space and $B \subseteq Y$ Borel. Let Γ be Σ^0_α or Π^0_α, $\alpha < \omega_1$. Then

B is $\overset{\Sigma}{\Gamma}$-hard $\iff B \notin \overset{\Pi}{\neg\Gamma}$.

To prove this, we first need an amusing lemma:

Lemma. For subsets A and B of 0-dim Polish spaces X and Y, either $(A, X) \leq_W (B, Y)$ or $(B, Y) \leq_W (A^c, X)$.

Proof. As we have shown in HW, X and Y are homeo to closed subsets of $\mathbb{N}^{\mathbb{N}}$, i.e. $X = [T]$ and $Y = [S]$.

We define the Wadge game $G_W(T, S, A, B)$:

P1. x_0 $\quad$ x_1 $\quad \dots$ $\qquad$ Rules: $(x_0, \dots, x_n) \in T$ and

P2. $\quad y_0$ $\quad$ y_1 $\qquad\qquad\qquad$ $(y_0, \dots, y_n) \in S$.

Player 2 wins $:\iff (x \in A \iff y \in B)$, where $x := (x_n)$, $y := (y_n)$.

$\iff (x, y) \in (A \times B) \vee (A^c \times B^c)$.

This is a Borel game, so it's determined. Suppose P2 has a winning strategy, which we may view as a monotone function $\varphi : T \to S$ which maps $(x_0, \ldots, x_n)$ to $(y_0, \ldots, y_n)$. Since $\forall\, t \in T$, $|\varphi(t)| = t$, so the induced map φ^* has domain $[T]$, so $\varphi^* : [T] \to [S]$ and is continuous.
Then φ^* is a reduction of A to B because if $x \in A$ then $\varphi^*(x)$ is the response of P2 to the play x of P1, hence $x \in A \iff \varphi^*(x) \in B$.

Similarly, if P1 has a winning strategy, then $B^c \leq_W A$. $\square$

Proof of Wadge's theorem. $\Rightarrow$. If B is Γ-complete, then $B \in \neg\Gamma$ would imply that $\Gamma \subseteq \neg\Gamma$, a contradiction.

$\Leftarrow$. Suppose that $B \notin \neg\Gamma$ and let $A \in \Gamma(X)$ for some 0-dim Polish X. By Wadge's lemma, $A \leq_W B$ or $B^c \leq_W A$. The latter option implies that $B^c \in \Gamma$ so $B \in \neg\Gamma$, a contradiction, so it must be that $A \leq_W B$. $\square$

We now build a complete analytic set. Each tree T on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ and hence $T \in 2^{(\mathbb{N}^{<\mathbb{N}})}$. The set Tr of trees is in fact a closed subset of $2^{(\mathbb{N}^{<\mathbb{N}})}$; indeed, for $T \in 2^{(\mathbb{N}^{<\mathbb{N}})}$,

$$T \in Tr \iff \underbrace{\forall s \in \mathbb{N}^{<\mathbb{N}},\ \underbrace{\underbrace{s \in T}_{\text{clopen}} \Rightarrow \underbrace{\forall i < n\ \ \underbrace{s|_i \in T}_{\text{clopen}}}_{\text{clopen}}}_{\text{clopen}}}_{\text{closed} \qquad \text{clopen}}.$$

Thus, Tr is a Polish space. Let IF denote the set of all ill-founded trees (i.e. containing an infinite branch) $T \in Tr$. Thus, $WF := Tr \setminus IF$ is the set of well-founded trees (no infinite branch).

Theorem. IF is complete-analytic. Hence WF is complete-coanalytic.

Proof. Let $A \subseteq$ be an analytic subset of a 0-dim Polish X.

X is homeo to a closed subset of $\mathbb{N}^{\mathbb{N}}$, so may assume $X \subseteq \mathbb{N}^{\mathbb{N}}$ closed so $A \subseteq \mathbb{N}^{\mathbb{N}}$ is analytic as a subset of $\mathbb{N}^{\mathbb{N}}$ because closed intersect analytic is still analytic.

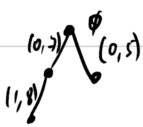
Thus, $A = \text{proj}_0 C$ where $C \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is closed.

Identifying $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \cong (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ and $C \subseteq (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$, we get a pruned tree T on $\mathbb{N} \times \mathbb{N}$ s.t. $C = [T]$.

Define $p : (\mathbb{N} \times \mathbb{N})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and note that $A = p[T] := p([T])$.

$$(x_n, y_n)_n \mapsto (x_n)_n$$

For each $x \in \mathbb{N}^{\mathbb{N}}$, let $T_x := \{ s \in \mathbb{N}^{<\mathbb{N}} : (x(i), s(i))_{i<|s|} \in T \}$.

E.g. 

$T_{01011\ldots} = \{(7), (5), (7,8)\}$, $T_{1000\ldots} = \{\emptyset\}$, $T_{00\ldots} = \{5\}$.

The map $f : \mathbb{N}^{\mathbb{N}} \to Tr$ by $x \mapsto T_x$ is continuous (why?), so it remains to show that it reduces A to IF.

$\forall x \in \mathbb{N}^{\mathbb{N}}$, $x \in A \iff x \in p[T] \iff \exists y \in \mathbb{N}^{\mathbb{N}}\ (x,y) \in [T]$
$\iff \exists y \in \mathbb{N}^{\mathbb{N}}\ y \in [T_x] \iff T_x \in IF$. $\square$

To show that a set B is not Borel, it's enough to continuously reduce IF to B. Not Borel means there is no ctbl algorithm determining the membership to B. This is loudly illustrated by the famous result of Foreman, Rudolph, and Weiss, which says that the old program of von Neumann on trying to classify all measure-preserving automorphism of $([0,1), \lambda)$ up to conjugacy is impossible:

Theorem (F-R-W). The conjugacy relation $\sim$ on the group $\mathrm{Aut}([0,1), \lambda)$ of all meas.-pres. autom. of $[0,1)$ is complete-analytic. In particular, $\nexists$ Borel function $\rho : \mathrm{Aut}([0,1), \lambda) \to \mathbb{R}$ s.t. $\forall \varphi, \psi \in \mathrm{Aut}([0,1), \lambda)$,

$$\varphi \sim \psi \iff \rho(\varphi) = \rho(\psi).$$

Proof of "in particular". O.w. let $\rho_2 : \mathrm{Aut}([0,1), \lambda)^2 \to \mathbb{R}^2$ by $(\varphi, \psi) \mapsto (\rho(\varphi), \rho(\psi))$ and $\sim = \rho_2^{-1}(\Delta_2)$, where Δ_2 is the diag of $\mathbb{R}^2$. Hence, $\sim$ is Borel, a contradiction. $\square$